ABSTRACT

This paper presents an optimal kinematic design for a general type of 3-RPS spatial parallel manipulator based on multi-objective optimization. The objective functions considered are Global Conditioning Index (GCI), Global stiffness Index (GSI) and Workspace volume. The objective functions are optimized simultaneously to improve the dexterity as well as the workspace volume which represents the working capacity of a parallel manipulator. A multi-objective Evolutionary Algorithm based on the control elitist non-dominated sorting genetic algorithm is adopted to find the true optimal Pareto front. A constraint Jacobian matrix is derived analytically and the manipulator workspace is generated by numerical search method. The static analysis of the manipulator is also carried out to determine the compliance of the end-effector.

Keywords: GCI; GSI; multi-objective genetic algorithm (MOO); Pareto front; workspace volume.
1. INTRODUCTION

Parallel manipulators have become popular in recent years due to the advantages in terms of high stiffness, high force-to-weight ratio, high load carrying capacity, and high precision control over the prescribed path of end-effector. A parallel manipulator typically consists of a moving platform that is connected to a fixed base by several legs in parallel. Gough and Whitehall [1] devised a six-linear jack system for use as Universal tire testing machine. The six degrees of freedom (DOF) Stewart platform has designed for use as an aircraft simulator by Stewart [2]. A systematic study of possible kinematic structures of parallel manipulators is carried out by K.H. Hunt [3]. The six DOF parallel manipulators have many advantages mentioned above; however six DOF is not always required for many practical applications. Spatial parallel manipulators with less than six DOF gained prominence because of reduction in manufacturing cost and easiness in control. Many 3-DOF parallel manipulators have been designed and investigated for specific applications, such as the famous DELTA robot with three translational DOF [4], the CaPaMan and HANA parallel manipulators with three spatial DOF [5,6]. A spatial 3-RPS parallel manipulator consists of three identical legs have been studied by Lee and Shah [7], Tsai [8] which allow the platform to rotate and translate. Joshi and Tsai [9] determined the singular positions of the 3-RPS manipulator through the Jacobian matrix. C.H. Liu et al. [10] applied a procedure to obtain direct singularity positions of a 3-RPS parallel manipulator, in which the heights of the three spherical joints were used as coordinate axes and the workspace of the moving platform represented as an inclined solid cylinder in this coordinate system. Yi Lu Bo Hu [11] proposed a unified and simple approach for solving inverse/forward velocity and acceleration of the limited-DOF PKMs with linear active legs. Jaime Gallardo et al. [12] have carried out the forward position analysis of parallel manipulators with identical limb type revolute-prismatic-spherical (R-P-S) leg by applying recursively the Sylvester dialytic elimination method. Rad et al. [13] addressed the analysis of forward kinematics of a 3-DOF medical parallel robot with R-P-S joint structure using Newton-Kantorovich (N-K) method, in which workspace was determined through forward kinematics equation. Stiffness is one of the most important aspects in the design of parallel kinematic machines because of higher stiffness allows higher feed rates with high accuracy in positioning of end-effector. Dan Zhang et al. [14] presented the kinetostatic modeling of the Tripod-based parallel kinematic machine and investigated the compliance over the workspace. Huy-Hoang Pham et al. [15] presented the stiffness matrices of a double linear spring and a three DOF translational flexure parallel mechanism. Joo-Woo Kim et al. [16] performed the stiffness analysis of a 3 DOF parallel Robot with one constraining leg, which takes into account of elastic deformation of joints and links which is made up of three SPS (spherical-prismatic-spherical) legs and one UP (Universal-prismatic) leg at the center. Qingsong Xu and Yangmin Li [17] analyzed the stiffness of a three prismatic-revolute-cylindrical (3PRC) translational parallel manipulator, in which stiffness matrix is derived based upon screw theory with consideration of compliance of both actuators and legs. Yi Lu and Bo Hu [18] explained the stiffness and elastic deformation for 3- 4 and 5-DOF PKMs with SPR type legs. D. Chablat et al. [19] presented a novel method for the stiffness analysis of over-constrained parallel manipulators. Many scholars have performed optimum design of robot manipulators; Boeig et al. [20] proposed numerical integration and sequential quadratic programming method for optimization of parallel manipulators. However, the traditional optimization methods suffer from local optima and lack of convergence of the optimization algorithm. Holland [21] has described how genetic algorithms may be applied as powerful and broadly applicable stochastic search methods and optimization techniques, since they can escape from local optima. Liu X-J et al. [22] introduce an approach to do optimum design of 3-DOF spherical parallel manipulators in order to optimize the performance indices GCI and GSI. Shiakolas et al. [23] used three methods (simple genetic algorithm, genetic algorithm with elitism and differential evolution) to get optimum design of 2-link and 3-link serial manipulators. Study of parallel manipulators with the optimization criteria as the manipulability or dexterity of the manipulator was done by Alice [24]. Marco Ceccarelli et al. [25] formu-
lated a multi-objective optimization problem for 3R serial link manipulators by taking the workspace volume and robot dimensions as objective functions for the given workspace limits as constraints. Liu et al. [26] presented a method for optimal kinematic design of a 3-dof parallel manipulator; the dimensional synthesis being carried out by introducing a tilt angle to achieve a nearly axial symmetry of kinematic performance with respect to configuration. N.M.Rao and K.M.Rao [27] have performed the dimensional synthesis of a 3-RPS parallel manipulator using a hybrid optimization method called GA-simplex method. Manuel R. Barbosa et al. [28] presented the kinematic design of a 6-DOF parallel manipulator for maximum dexterity, using the GA and neuro-GA in order to explore the advantages of neural networks and GA’s. Raza Ur-Rehman et al. [29] propose a methodology to deal with the multi-objective design optimization of 3-PRR planar parallel manipulator with the size of the regular shaped workspace and the mass in motion of the mechanism are as the objective functions. F.A. Lara-Molina et al. [30] performed the optimal design of a spatial Stewart-Gough platform based on multi-objective optimization; the objective functions are Global Condition Index (GCI), Global Payload Index (GPI) and Global Gradient Index (GGI) using multi-objective Evolutionary Algorithm (MOEA). Chun-Ta Chen et al. [31] presented a constrained multi-objective genetic algorithm for a general motor-driven parallel kinematic manipulator 3UPS for the optimal trajectory of a PKM with linear actuators, in which travelling time and energy expended in driving the platform from one pose to another are minimized. Antonio M. Lopes and E.J. Solteiro Pires [32] formulated an optimization problem in order to minimize power consumption and maximize the stiffness of the manipulator using a multi-objective genetic algorithm in order to find the work piece location in a machining Robotic cell. Ridha Kelaiaia et al. [33] presented an approach for dimensional synthesis for parallel manipulators. The problem is defined as a multi-objective optimization in order to maximize the performance (Workspace, stiffness, kinematic, dynamic performance) using the strength Pareto Evolutionary algorithm-II (SPEA-II). A constrained Jacobian matrix in three independent end-effector position variables both in vector form and analytical form is investigated in this paper. This Jacobian matrix is useful in measuring the performance of the manipulator. It is necessary to analyze kinematic characteristics of a 3RPS manipulator from the viewpoints of conceptual design and engineering control. The size and shape of the workspace volume of the 3RPS manipulator is studied in detail based on the inverse kinematic analysis rather than the direct kinematics in which all possible configurations at most sixteen solutions of the moving platform are obtained for a given set of inputs however, it is time-consuming and also some solutions may not represent the actual configuration. The second objective of this paper is to carry out the stiffness analysis for a 3RPS manipulator based on the concept developed by earlier researchers which is a general method for stiffness analysis of some 3–5 DOF S-P-R spatial parallel manipulators. The actuation forces of the prismatic joints and the compliance of the end-effector for given end-effector trajectory is enumerated. A multi-objective Genetic Algorithm problem is formulated for obtaining the optimal geometric parameters by optimizing the performance indices GCI, GSI and Workspace volume using the controlled elitist non-dominated sorting genetic algorithm. The organization of the paper is as follows: In Section 2 is described the geometry of the manipulator and the kinematic analysis of mechanism. Section 3 explains the derivation of Jacobian matrix and significance of Global Condition Index. The reachable work volume is in shown in Section 4, and the stiffness analysis is presented in Section 5. Multi-objective optimization is described in Section 6. Finally, this paper concludes with a discussion of scope of further research in Section 7.

2. KINEMATIC ANALYSIS

2.1. Geometric Description and Position Analysis of Manipulator

Figure 1 shows the geometry of the spatial 3-RPS parallel manipulator, in which the moving platform B has three spherical joints S located at the vertices of the equilateral triangle $B_i$ ($i = 1, 2, 3$). The base
platform $A$ has three revolute joints $R$ at the corners of the equilateral triangle $A_i \ (i = 1, 2, 3)$. The fixed base and the moving platform are connected by means of prismatic joints.

The axes of revolute joints are $J_i (i = 1, 2, 3)$, $a_i = [a_x \ a_y \ a_z]^T$ is the position vector of the location revolute joint $A_i$ with respect to the fixed (reference) frame $XYZ$ which is located at $O$ which is the centered point of the fixed base platform, and $b_i = [b_x \ b_y \ b_z]^T$ is the position vector of the spherical joint $B_i$ with respect to the moving frame “uvw” which is located at $P$ which is the centered point of moving platform. The direction of each revolute joint with respect to the fixed frame is indicated by a unit vector along its axis $J_i = [j_x \ j_y \ j_z]^T$. The magnitude of the position vector $a_i$ (size of the fixed platform) is represented by $g$ and the magnitude of the position vector $b_i$ (size of the moving platform) is represented by $h$. The $i^{th}$ leg length and unit vector are represented by $d_i$ and $w_i$ respectively.

The transformation from the moving frame to the fixed frame can be described by a position vector $p = OP$ and a $3 \times 3$ rotation matrix $O R P$. Let $u$, $v$ and $w$ be three unit vectors defined along $u$, $v$ and $w$ axes of the moving frame respectively; then the rotation matrix can be expressed in terms of the direction cosines of $u$, $v$ and $w$ as:

$$O R P = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}. \quad (1)$$

The three Euler angles $\gamma, \alpha, \beta$ rotating about the $Z$, $X$ and $Y$-axes of the fixed reference frame in sequence that is (Y-X-Z) Euler angle system is considered, the transformation from moving frame to the fixed frame is given by:

$$O R P = R_y(\beta) R_x(\alpha) R_z(\gamma) = \begin{bmatrix} c\beta c\gamma + s\alpha s\beta s\gamma & -c\beta s\gamma + s\alpha c\beta c\gamma & c\alpha c\beta \\ c\alpha s\gamma & c\alpha c\gamma & -s\alpha \\ -s\beta c\gamma + s\alpha c\beta s\gamma & s\beta s\gamma + s\alpha c\beta c\gamma & c\alpha c\beta \end{bmatrix}. \quad (2)$$

The coordinates of $A_i$ and $B_i$ are given by:

$$a_1 = \begin{bmatrix} g & 0 & 0 \end{bmatrix}^T, \quad a_2 = \begin{bmatrix} -\frac{1}{2} g & -\frac{\sqrt{3}}{2} g & 0 \end{bmatrix}^T, \quad a_3 = \begin{bmatrix} -\frac{1}{2} g & -\frac{\sqrt{3}}{2} g & 0 \end{bmatrix}^T. \quad (3)$$
$$P_{b_1} = \begin{bmatrix} h & 0 & 0 \end{bmatrix}^T, \quad P_{b_2} = \begin{bmatrix} -\frac{1}{2}h & \frac{\sqrt{3}}{2}h & 0 \end{bmatrix}^T, \quad P_{b_3} = \begin{bmatrix} -\frac{1}{2}h & -\frac{\sqrt{3}}{2}h & 0 \end{bmatrix}^T. \quad (4)$$

The position vector $q_i$ of point $B_i$ from the reference point can be expressed as:

$$q_i = P + O R P P_{b_i}.$$  \quad (5)

Substituting Eq. (2) and Eq. (4) in Eq. (5) yields:

$$\begin{bmatrix} p_x + hu_x \\
p_y + hu_y \\
p_z + hu_z \end{bmatrix}, \quad \begin{bmatrix} p_x - \frac{hu_x}{2} + \frac{\sqrt{3}hv_x}{2} \\
p_y - \frac{hu_y}{2} + \frac{\sqrt{3}hv_y}{2} \\
p_z - \frac{hu_z}{2} + \frac{\sqrt{3}hv_z}{2} \end{bmatrix}, \quad \begin{bmatrix} p_x - \frac{hu_x}{2} - \frac{\sqrt{3}hv_x}{2} \\
p_y - \frac{hu_y}{2} - \frac{\sqrt{3}hv_y}{2} \\
p_z - \frac{hu_z}{2} - \frac{\sqrt{3}hv_z}{2} \end{bmatrix}. \quad (6)$$

Considering the mechanical constraints imposed by a revolute joint located at $A_i$ its motion is constrained in one of the following three planes:

$$q_{1y} = 0, \quad q_{2y} = -\sqrt{3}q_{2x}, \quad q_{3y} = +\sqrt{3}q_{3x}. \quad (7-9)$$

Substituting the $y$-components of $q_i$ from Eq. (6) into Eqs. (7–9) yields:

$$p_y + hu_y = 0, \quad (10)$$

$$p_y - \frac{1}{2}hu_y + \frac{\sqrt{3}}{2}hv_y = -\sqrt{3}(p_x - \frac{1}{2}hu_x + \frac{\sqrt{3}}{2}hv_x), \quad (11)$$

$$p_y - \frac{1}{2}hu_y - \frac{\sqrt{3}}{2}hv_y = \sqrt{3}(p_x - \frac{1}{2}hu_x - \frac{\sqrt{3}}{2}hv_x). \quad (12)$$

Taking $2 \times$ Eqs. (10–12) yields:

$$v_x = u_y. \quad (13)$$

Subtracting Eq. (11) from Eq. (12) leads to:

$$p_x = \frac{h}{2}(u_x - v_y). \quad (14)$$

The relation between the output variables given by the three Eqs. (10,13,14) constrain the motion of the moving platform. Substituting the components of unit vectors $u$ and $v$ expressed in Eq. (2) into the three constraint Eqs. (10,13,14) yields:

$$-hc\alpha y = p_y, \quad (15)$$

$$-c\beta sy + s\alpha c\beta cy = c\alpha s\gamma y, \quad (16)$$

$$\frac{h}{2}(c\beta cy + s\alpha \beta sy - c\alpha \gamma y) = P_x. \quad (17)$$

Solving Eq. (16), we get:

$$\lambda = \tan^{-1}2(s\alpha \beta, c\alpha + c\beta). \quad (18)$$
2.2. Constraint Jacobian Analysis

The position vector of the ball joint \( B_i \) is expressed as:
\[
q_i = p + b_i .
\]
(19)
\[
q_i = a_i + d_i w_i ,
\]
(20)
where \( d_i \) and \( w_i \) are the length and unit vector of the \( i^{th} \) leg respectively.

Differentiating Eq. (19) with respect to time gives velocity of the centre of a ball joint \( B_i \):
\[
v_{Bi} = v_p + \omega_p \times b_i .
\]
(21)
Differentiating Eq. (20) with respect to time for the \( i^{th} \) limb, yields the velocity of the centre of a ball joint \( B_i \), which can be expressed as:
\[
v_{Bi} = d_i \omega_i \times w_i + \dot{d}_i w_i .
\]
(22)
where \( \omega_i \) is the angular velocity of the \( i^{th} \) leg, the above expressions Eqs. (21, 22) both represent the velocity of the ball joint \( B_i \), so equate them as:
\[
d_i \omega_i \times w_i + \dot{d}_i w_i = v_p + \omega_p \times b_i .
\]
(23)
Dot multiplying both sides of the Eq. (23) with \( w_i \) yields:
\[
\dot{d}_i = w_i \cdot v_p + (w_i \times b_i) \cdot \omega_p .
\]
(24)
Equation (24) can be written in the matrix form as:
\[
J_x \dot{X} = J_q \dot{d} .
\]
(25)
where
\[
J_x = \begin{bmatrix} w_1^T (w_1 \times b_1)^T \\ w_2^T (w_2 \times b_2)^T \\ w_3^T (w_3 \times b_3)^T \end{bmatrix} ;
J_q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ;
\dot{X} = [v_p ; \omega_p]^T , \dot{d} = [\dot{d}_1 \, \dot{d}_2 \, \dot{d}_3]^T .
\]

The above can be expressed as:
\[
\dot{d} = J_a \dot{X} .
\]
(26)
where \( J_a = J_q^{-1} J_x \) is a \( 3 \times 6 \) matrix.

It should be noted that the six linear and angular velocity components in \( \dot{X} \) are not all independent since 3-RPS parallel manipulator possesses only three degrees of freedom. Let \( p_z, \alpha, \) and \( \beta \) be the specified independent variables. Let \( \dot{x} = [\dot{p}_z \, \dot{\alpha} \, \dot{\beta}]^T \) when the manipulator is away from singularities, we have:
\[
\dot{X} = J_r \dot{x} .
\]
(27)
where
\[
J_r = \begin{bmatrix} \frac{\partial p_x}{\partial p_z} & \frac{\partial p_x}{\partial \alpha} & \frac{\partial p_x}{\partial \beta} \\ \frac{\partial p_y}{\partial p_z} & \frac{\partial p_y}{\partial \alpha} & \frac{\partial p_y}{\partial \beta} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial \gamma}{\partial p_z} & \frac{\partial \gamma}{\partial \alpha} & \frac{\partial \gamma}{\partial \beta} \end{bmatrix} .
\]
(28)
The vector loop closure equation for $i^{th}$ limb from the geometry of the mechanism is given as:

$$d_i = p + O R_p P b_i - a_i.$$  \hspace{1cm} (29)

Taking the dot product on both sides of the above equation yields:

$$d_i^2 = [q_i - a_i]^T [q_i - a_i] \quad for \quad i = 1, 2, 3.$$  \hspace{1cm} (30)

In order to get the analytical constrained Jacobian which transforms the joint velocity rates into the end-effector velocity vector defined by $[\dot{\alpha} \dot{\beta} \dot{p}_z]^T$ the above equation Eq. (30) has to be differentiated with respect to time and is expressed in matrix form as:

$$
\begin{bmatrix}
  d_1 & 0 & 0 \\
  0 & d_2 & 0 \\
  0 & 0 & d_3
\end{bmatrix}
\begin{bmatrix}
  \dot{d}_1 \\
  \dot{d}_2 \\
  \dot{d}_3
\end{bmatrix}
= 
\begin{bmatrix}
  m_1 & n_1 & l_1 \\
  m_2 & n_2 & l_2 \\
  m_3 & n_3 & l_3
\end{bmatrix}
\begin{bmatrix}
  \dot{\alpha} \\
  \dot{\beta} \\
  \dot{p}_z
\end{bmatrix}.
$$  \hspace{1cm} (31)

Equation (31) is constrained Jacobian in analytical form, the elements of the matrix has given in Appendix.

### 3. WORKSPACE ANALYSIS

The workspace of a parallel manipulator is one of the most aspect because of it reflects its working capacity. So it is necessary to analyze shape and size of the workspace volume for enhancing the applications of parallel manipulators. While designing a practical manipulator the physical constraints intern of the range of parasitic motions of the end-effector and limits of the linear actuators, leg interferences and limitations on the passive joints are to be considered. C. Gosselin [34], J.P. Merlet [35] presented an algorithm enabling to compute the possible rotation of the end-effector around a fixed point.

#### 3.1. Algorithm for the Workspace Volume

- **Step 1:** Initialize double arrays $A$, $B$ and $Z$, with dimensions $(n + 2) \times m$, where $n$ is the number of equally spaced planes $z$ between $z_{\text{min}}$ and $z_{\text{max}}$ at which the workspace will be computed, and $m$ is the number of points to be computed at each plane $z = \text{constant}$. These arrays will store, respectively, the values of $\gamma$, $\beta$ and $z$ for the points defining the workspace boundary.

- **Step 2:** Set $g$, $h$ and $d_{i_{\text{max}}}$, $d_{i_{\text{min}}}$ for $i = 1, 2, 3$

- **Step 3:** $z = z_{\text{min}}$

- **Step 4:** For the current $z$, construct the polar coordinate system at $(\gamma, \beta)$. Starting at $m$ equally spaced angles, increment the polar ray, solve the inverse kinematics and apply the constraint checks defined by Eqs. (30–32) until a constraint is violated. The values for $\gamma$, $\beta$ and $z$ at the point of constraint violation is written into the three double arrays.

- **Step 5:** Set $z = z + \Delta z$ where $\Delta z = \frac{z_{\text{max}} - z_{\text{min}}}{n}$

- **Step 6:** Repeat steps 4 and 5 until $z$ becomes greater than $z_{\text{max}}$.

- **Step 7:** Transfer $A$ and $B$ into $X$ and $Y$, so that $X(i, j) = B(i, j) \cos[A(i, j)]$ and $Y(i, j) = B(i, j) \sin[A(i, j)]$ where $i = 1 \ldots n + 1$ and $j = 1 \ldots m$. 

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The researchable workspace volume can be calculated by:

$$V = \sum_{Z_i = Z_{\text{min}}}^{Z_{\text{max}}} \sum_{\theta_z = 0}^{2\pi} \left( \frac{1}{2} r_i^2 \Delta \theta_z \Delta z \right),$$

(32)

where $r_i$ is polar ray, $\Delta \theta_z$ is the angle of increment when rotating about $Z$ axis and $\Delta z$ is small increment along the $Z$ axis.

### 3.2. Workspace Simulation

Since symmetric architectures are commonly considered in literature, 3RPS manipulator with both $\Delta A_1 A_2 A_3$ and $\Delta B_1 B_2 B_3$ as equilateral triangles with $|OA_1| = |OA_2| = |OA_3| = g = 2$ m, ($|PB_1| = |PB_2| = |PB_3| = h = 1$ m) and has been taken up as an example. The possible maximal leg lengths range, for each leg $i = 1, 2, 3$ considered here is $[1 \text{ to } 4]$ m and the two rotational Euler angles $\alpha, \beta$ of the moving platform rotating about the $X$ and $Y$ axes of the fixed reference frame respectively are bounded in the range $[-120^\circ \text{ to } 120^\circ]$. It is necessary to analyze the shape and size of the work volume for enhancing the application of parallel manipulators. The reachable workspace volume of 3RPS manipulator is shown in Fig. 2, which is plotted using the inverse kinematic solutions instead of the multiple solutions offered by the direct kinematics. The work volume is continuous and there are no vertical gaps along the $z$-coordinate, the middle portion of the reachable workspace volume is cylindrical in shape and the top portion is in the shape of a pyramid. Larger workspace size for the middle sections and smaller workspace size for top sections of the workspace volume is observed. The workspace at various cross-sections is plotted, Fig. 3 shows the workspace at $p_z = 0.5$ m is the bottom section of the workspace volume. In this it is observed that workspace is clearly divided into two parts due the presence of direct kinematic (internal) singularities and is viewed as nearly circular in shape. These singularities arise when the normal vectors to plane containing points $A_i$, $B_i$ and $P$ are parallel to the fixed platform and legs are aligned with the moving platform. The workspace at $p_z = 1.3$ m is shown in Fig. 4; it appears more like a circle representing the boundary singularities. The workspace at $p_z = 2.3$ m at the top most portion of the work volume shown in Fig. 5 is viewed as triangular with lobbed sides.

![Fig. 2. Workspace volume of 3RPS parallel manipulator.](image)
Fig. 3. Sectional workspace at $p_z = 0.5$ m.

Fig. 4. Sectional workspace at $p_z = 1.3$ m.
4. STIFFNESS ANALYSIS OF 3RPS MANIPULATOR

A wrench \([F \ M]\) applied at the centre point of the moving platform, where \(F\) is a concentrated force and \(M\) is a moment. The external wrench is balanced by the active forces \(F_{a_i} (i = 1, 2, 3)\) which acts along \(d_i\) and a constrained force \(F_{c_i} (i = 1, 2, 3)\) which acts at \(B_i\) passing through the spherical joint and parallel to the \(R_i\) \((i = 1, 2, 3)\) the revolute joint at the base platform as shown in Fig. 1. Based on the principle of virtual work the work done by a vector of active/constrained forces \(F_d = \begin{bmatrix} F_{a1} & F_{a2} & F_{a3} & F_{c1} & F_{c2} & F_{c3} \end{bmatrix}^T\) be the same as that of work done by the wrench \([F \ M]\) which is expressed as:

\[
\dot{X}F_d^T \dot{d} + \begin{bmatrix} F \\ M \end{bmatrix}^T = 0,
\]

(33)

\[
\begin{bmatrix} F \\ M \end{bmatrix} = -J_e^T F_d.
\]

(34)

where \(\dot{d} = \begin{bmatrix} \dot{d}_1 & \dot{d}_2 & \dot{d}_3 & 0 & 0 & 0 \end{bmatrix}^T\).

\[
J_e = \begin{bmatrix}
   w_1^T (b_1 \times w_1)^T \\
   w_2^T (b_2 \times w_2)^T \\
   w_3^T (b_3 \times w_3)^T \\
   c_1^T (b_1 \times c_1)^T \\
   c_2^T (b_2 \times c_2)^T \\
   c_3^T (b_3 \times c_3)^T
\end{bmatrix}.
\]
When an active force $F_{ai}$ is applied on the RPS leg along $d_i$, the longitudinal elastic deformation $\delta d_{ai}$ of leg $d_i$ is shown in Fig. 6(a) can be given by Timoshenko and Gere [36] as:

$$\delta d_{ai} = \frac{F_{ai}}{k_{ai}}, \text{ where } k_{ai} = \frac{-E_i}{\left(\frac{d_i - d_{ii}}{B_{ii}} + \frac{d_{ii}}{B_{2i}}\right)}.$$  \hspace{1cm} (35)

When a constrained force $F_{ci}$ is exerted on a RPS leg $d_i$ at the spherical joint $B_i$ and is parallel to the axis of the revolute joint ($F_{ci} \parallel R_i$), the transverse elastic deformation $\delta d_{bi}$ is shown in Fig. 6(b) and can be solved as:

$$\delta d_{bi} = \frac{F_{ci}}{k_{ci}}, \text{ where } k_{ci} = \frac{-3E_i}{\left(\frac{d_{ii}^2}{T_{ii}} + \frac{(d_i - d_{ii})(d_{ii}^2 + d_{ii}^2 + d_{ii}^2)}{I_{2i}}\right)},$$ \hspace{1cm} (36)

where $k_{ai}$ and $k_{ci}$ are longitudinal and transverse stiffness of the RPS type leg respectively. The force and deformation relations can be expressed in matrix form using the above two Eqs. (35,36) as:

$$F_d = K \begin{bmatrix} \delta d_{a1} \\ \delta d_{a2} \\ \delta d_{a3} \\ \delta d_{b1} \\ \delta d_{b2} \\ \delta d_{b3} \end{bmatrix}, \text{ where } K = \begin{bmatrix} k_{a1} & 0 & 0 & 0 & 0 \\ 0 & k_{a2} & 0 & 0 & 0 \\ 0 & 0 & k_{a3} & 0 & 0 \\ 0 & 0 & 0 & k_{c1} & 0 \\ 0 & 0 & 0 & 0 & k_{c2} \\ 0 & 0 & 0 & 0 & k_{c3} \end{bmatrix}.$$ \hspace{1cm} (37)

Fig. 6. Elastic deformation of RPS leg: (a) axial deformation and (b) transverse deformation.

### 4.1. Analytic Solved Example

Let $d_{1i}$, $B_{1i}$ and $I_{1i}$ be the length, section and moment of inertia of the piston of leg $d_i$ respectively. Let $(d_i - d_{1i})$, $B_2i$ and $I_2i$ be the length, section and moment of inertia of a cylinder of the leg $d_i$ respectively. $E_i$ be the modular of elasticity for leg $d_i$. In the 3RPS manipulator the independent pose variables $(p_z, \alpha, \beta)$ vary with time $t$. For the given parameters $g = 2 \text{ m}, h = 1 \text{ m}, 1 \leq d_i \leq 4 \text{ m}, E_i = 2.11 \times 10^{11} \text{ N/m}^2, E_i I_{1i} = E_i I_{i2} = 26.800 \text{ N} \cdot \text{m}^2, B_{1i} = B_{12} = 0.0015 \text{ m}^2, F = [-20, -20, -20] \text{ KN},$
$T = [-80, -80, -80]$ KN–m, the actuation forces ($F_{ai}$) of the three active legs ($d_i$) for the given end-effector trajectory $p_x = 0$; $p_y = 0$; $p_z = z_0 + 0, 1t$ is plotted as shown in Fig. 7. It is observed that the actuating forces are gradually decreasing with increasing time; the actuating forces are lower at higher values of $p_z$ and higher at low $p_z$ values. The lower magnitudes of forces are registered for the leg 1 and higher magnitudes of forces for leg 3 when comparing the three actuators.

![Fig. 7. Variation of actuating forces for 3RPS manipulator.](image)

The vertical deflections along the $z$-coordinate is linearly increasing with time, higher values of vertical deflections are obtained at the higher $z$-coordinates from the fixed reference frame as shown in Fig. 8, because of the stiffness is largest when the platform is located at its initial pose at which the extensions of the active legs are smaller. When the moving platform is away from the initial pose the extensions of the active legs are higher so the deflections are also higher values. The deflections along the $x$-coordinate and $y$-coordinate is shown in Fig. 9, the deflections are also increasing with increase of time for the given end-effector trajectory. These deflections are considerable and which greatly influence the accuracy of the manipulator, these values have to be minimized.

### 5. KINEMATIC PERFORMANCE INDEX

Dexterity is an important issue for design, trajectory planning, and control of manipulators. The dexterity of a manipulator can be thought as the ability of the manipulator to arbitrarily change its position and orientation, dexterity is measured in terms of the Jacobian matrix. The two Global performance indices GCI, GSI are widely adopted for the evaluation of global behavior of manipulators.

#### 5.1. Global Conditioning Index

The condition number of a Jacobian matrix is expressed as:

$$\kappa = \frac{\|J^{-1}\|}{\|J\|}.$$  (38)
Condition number signifies the error amplification factor between the joint rate errors to task space errors. The condition number depends on the manipulator configuration. The condition number varies from one at isotropic configurations to infinity at singular configurations so this is also defined as the measure of degree
of ill-conditioning of the Jacobian matrix. The reciprocal of the condition number \(1/\kappa\) is referred to as the conditioning index which is the local measure of performance of the manipulator at a particular pose of the end-effector within in the workspace. In order to evaluate the global behavior of a manipulator over the workspace, a global index is proposed by Gosselin and Angeles [37] as:

\[
GCI = \frac{\int_{W} (\frac{1}{\kappa}) dW}{\int_{W} dW},
\]

(39)

where \(W\) is the workspace, \(\kappa\) is the condition number.

In other sense GCI is average value of \(1/\kappa\) over workspace region; it represents the uniformity of dexterity over the entire workspace so this index makes sense for the optimal design of manipulator for which the average value performance is an important design factor.

5.2. Global Stiffness Index

If an external wrench exerts on the moving platform, there is a deformation along different directions which can regarded as being distributed on an ellipsoid sphere with the lengths of major axis and minor axis being the maximum and minimum value of the deflection, respectively. This deformation is dependent on the manipulator’s stiffness and on the external wrench. The manipulator stiffness affects the dynamics and position accuracy of the device, for which stiffness is considered as an important performance index. In spatial coordinate system the stiffness matrix \(K\) that relates the external wrench vector \(\tau\) at the moving platform to the output displacement vector \(D\) according to:

\[
D = K^{-1}\tau,
\]

(40)

where the stiffness matrix \(K\) is expressed as:

\[
K = J^T K_p J,
\]

(41)

in which \(K_p = diag(k_{p1}, k_{p2}, k_{p3})\).

Let \(k_{p1} = k_{p2} = k_{p3} = 1\) and \(\|\tau\| = 1\), the maximum and minimum deformations can be obtained as:

\[
\|D_{\text{max}}\| = \sqrt{\max(|\lambda_{Di}|)} \quad \text{and} \quad \|D_{\text{min}}\| = \sqrt{\min(|\lambda_{Di}|)}.
\]

(42)

where \(\lambda_{Di}\) (\(i = 1, 2, 3\)) are the Eigen values of the matrix \((K^{-1})^TK^{-1}\).

Similarly the global stiffness index that can evaluate the stiffness of the mechanism with the workspace is briefed by Xin-Jun Liu et al. [38] as:

\[
\eta_{D_{\text{max}}} = \frac{\int_{W} \|D_{\text{max}}\| dW}{\int_{W} dW} \quad \text{and} \quad \eta_{D_{\text{min}}} = \frac{\int_{W} \|D_{\text{min}}\| dW}{\int_{W} dW}.
\]

(43)

In practical application, it is expected that the minimum deformation should be smaller. The smaller the minimum deformation is, the better the stiffness performance of a manipulator is.

6. GENETIC ALGORITHMS

GA is a Meta heuristic search algorithm that uses a population of designs rather than a single design at a time and utilizes the concepts of natural selection and survival of the fittest among biological structures.
An initial randomized population that consists of a group of chromosomes which represent the problem variables, produces new populations through successive iterations, using various genetic operators. The common operators are selection, mutation, and crossover. Each chromosome is a potential solution and is comprised of a series of substrings or genes which can be used to evaluate the objective function of the problem. The algorithm is run through several iterations, called generations, and a fitness function is used to guide the search for solutions through the search space.

6.1. Multi-Objective Optimization

A multi-objective optimization problem has a number of objective functions which are to be minimized or maximized. The general form of the multi-objective minimization problem is as:

\[
\text{Minimize } F(x) = (f_1(x), f_2(x), \ldots, f_i(x)) \quad (i = 1, 2 \ldots M) \\
\text{Subjected to } g_j(x) \leq 0 \quad (j = 1, 2 \ldots N) \\
h_k(x) = 0 \quad (k = 1, 2 \ldots L)
\]

A solution \( x \) is a vector of \( n \) decision variables \( x = (x_1, x_2, \ldots, x_n) \). The component objective functions \( f_1(x), f_2(x), \ldots, f_m(x) \) are competent with each other for solution, usually there is no single solution for which all objectives are optimal simultaneously and none of these solutions can be said to be better than the other with respect to all the objectives, these solutions are called non-dominated solutions also called as Pareto optimal solutions. This particular set has a property of dominating all other solutions which are not belonging to this set. Thus, each solution is important with respect to some trade-off relationship among the objectives. A curve formed by joining all these solutions is known as Pareto-optimal front. Deb et al. [39] proposed the NSGA-II algorithm, which is the most preferred algorithm to solve the multi-objective problems using the principles of GA. NSGA-II not only emphasizes on non-dominated solutions but also simultaneously maintains diversity in the non-dominated solutions using an elite-preserving mechanism.

6.2. Controlled Elitist Non-Dominated Sorting Genetic Algorithm

The Multi-Objective Evolutionary Algorithm preferred to solve this optimization problem is the Controlled Elitist Non-dominated Sorting Genetic Algorithm (CENSGA), a variant of NSGA-II was proposed by Deb and Goel [40] for controlling the extent of the elite members of the population and to maintain the diversity of the population for convergence to an optimal Pareto front. The elitism is controlled by a geometric decreasing function in which a user defined parameter \( r \) (reduction rate) plays an important role in determining the maximum number of allowed individuals in each non-dominated front. According to the geometric distribution each front is allowed to have an exponentially reducing number of solutions. The main concept of the CENSGA is to forcibly allow at least certain number of solutions from each non-dominated fronts to co-exist in the new population. The selection of specified number of solutions from each front is achieved by crowded tournament selection. The concept of CENSGA is almost same as that of NSGA-II except the controlled elite preserving mechanism, the new population obtained in CENSGA will be more diverse than that obtained by using the NSGA-II.

6.3. Simulation of GA

Four simulations are performed using the Matlab optimization toolbox. The first simulation is a single objective optimization to maximize the global conditioning index, the second and third simulations are also single objective problems to maximize the GSI, Workspace volume respectively. The fourth simulation is a multi-objective optimization to optimize the GCI, GSI, Workspace volume simultaneously.
6.4. Single-Objective GA

The design vector $x = \begin{bmatrix} h & g \end{bmatrix}$ is optimized for obtaining the Maximum GCI in the first simulation; in this case the optimization problem is formulated as:

$$\text{Max} \{ GCI(x) \},$$

Subjected to:

$$h + g + d_0 = \lambda \quad \text{(constraint on the manipulator size)}, \quad (44)$$

$$d_{i \text{ min}} \leq d_i \leq d_{i \text{ max}} \quad \text{for} \ i = 1, 2, 3,$$

$$0.5 \leq h \leq 2.5,$$

$$0.5 \leq g \leq 2.5 \quad \text{(boundary constraints)},$$

where $d_0$ is the leg length at home position (where all legs are half actuated) and $\lambda$ a constant representing the relative size of the manipulator. For a given leg length $d_i$ bounded $[1 \ 3]$ m, and the three rotational Euler angles $\gamma, \alpha, \beta$ of the moving platform rotating about the $Z, X$ and $Y$-axes of the fixed reference frame respectively are bounded in the range $[-60^\circ \ 60^\circ]$. In the first simulation of single objective GA, GCI is evaluated with the GA parameters given in Table 1 and the plotted results are shown in Fig. 10. Generally Genetic algorithms are best suited for minimization problems so the fitness function has been considered here is the reciprocal of the GCI. The optimal fitness (GCI) value has been found to be 0.5601 and the corresponding design vector $x = \begin{bmatrix} h & g \end{bmatrix} = [1.4166 \ 1.0825]$. 

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Population size</th>
<th>Maximum generations</th>
<th>Encoding type</th>
<th>Selection strategy</th>
<th>Crossover type</th>
<th>Mutation type</th>
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Table 1. Simple GA parameters.

The global stiffness index (GSI) is maximized in the second simulation run, subjected to the same constraints mentioned in Eq. (44) and for the same GA parameters given in Table 1. The evolution of GSI as a function of generations is shown in Fig. 11, the best value $GSI = 0.2102$ is converged in 52 generations and the corresponding design vector $x = [1.5043 \ 0.9947]$. In the third GA simulation run workspace volume is maximized for the same set of constraints given in Eq. (44) and for same GA parameters which are given in Table 1. The evolution of workspace volume as a function of generations is shown in Fig. 12, the best value for workspace volume $= 5.0352 \ m^3$ is converged in 57 generations for the design parameters $x = [1.8707 \ 0.6303]$. The design vector $x = [h \ g]$ is not same in all the three single objective simulation runs for the evolution of Maximum fitness values, so it is observed that there is a contradictory among the objectives for their optimal values.
Fig. 10. Variation of GCI over the generations.

Fig. 11. Variation of GSI over the generations.
6.5. Multi-Objective Optimization

A multi criteria design objective function is defined as:

$$\text{Max} \{ \text{GCI, GSI and Workspace volume} \}.$$ \hspace{1cm} (45)

Subjected to the same constraints given in Eq. (43). After some preliminary simulations, we set simulation parameters to run the CENSGA is given in Table 2. The three dimensional Pareto front in criterion space is shown in Fig. 13, and some non-dominated solutions of the Pareto front representing the three objectives (GCI, GSI and Workspace volume) given in Table 3. The Pareto front solution in decision space is given Table 4, which corresponds directly to the set of optimal design parameters of the manipulator. We can observe that how the function values vary along the Pareto front, in Fig. 14 the convex Pareto front between GCI and workspace volume in which it is observed that when the dexterity increases, the workspace volume decreases. The ideal solution would be maximizes both the functions so an intermediate solution is to be chosen depending on the designer’s choice. The convex Pareto front between GSI and workspace volume is shown in Fig. 15, here also it is observed that GSI is increased with decrease in workspace volume. In Fig. 16, it is observed that increasing of GCI also increases the GSI.

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Table 2. CENSGA algorithm parameters.
Fig. 13. Three dimensional MOO Pareto front in criterion space.

<table>
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Table 3. Pareto front, criterion space.

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<th>4</th>
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<th>6</th>
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<th>8</th>
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<td>0.5675</td>
<td>0.2626</td>
<td>0.5608</td>
<td>0.5675</td>
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<td>GSI</td>
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<td>1.8688</td>
<td>0.6564</td>
<td>3.3967</td>
<td>4.7574</td>
<td>2.0296</td>
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Table 4. Pareto front, decision space.

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Fig. 14. Pareto front GCI and workspace.

Fig. 15. Pareto front GSI and workspace.
7. CONCLUSION

The architecture optimization for a general 3-RPS parallel manipulator is performed by optimizing the two global kinematic performance indices and workspace volume simultaneously which are of contradictory functions have to be satisfied. The CENSGA is a robust optimization tool provides a true optimal Pareto front with widely spread non-dominated solutions. The described approach is absolutely generic and can be used with different objective functions and constraints. A constrained Jacobian matrix is derived and the reachable workspace is generated using numerical search method. The stiffness analysis is presented in details and stiffness matrix is derived based upon an overall Jacobian matrix.

REFERENCES

A. APPENDIX

Where:

\[ m_1 = -\frac{h^2}{8} s_2 \alpha s_2 y c \beta + \frac{h^2}{2} c^2 \gamma c \alpha s_2 \beta + \frac{h^2}{4} s^2 y s_2 \alpha s_2 \beta + \frac{h^2}{4} s_2 y c \beta s_2 \alpha \]

\[ -2h p_2 s_2 a s y c \beta + g h c y s \beta - h^2 c y s \beta + h^2 s y s a c \beta - 2h c y s \beta + 2 h s y s a c \beta \]

\[ -2h p_2 c y c \beta - g h s a s y c \beta + 2 g h c y s \beta - 2 g h s a s y c \beta + k_1 (2z s \alpha + z s_2 a c \beta) \]

\[ n_1 = \left[ + \frac{h^2}{2} c^2 \gamma c \beta s \alpha h^2 s_2 y s^2 y c s_2 \alpha - \frac{h^2}{4} c^2 y s_2 \alpha + \frac{h^2}{8} s^2 \beta s_2 y c \alpha - \frac{h^2}{4} s_2 y c \alpha s_2 \beta + 2h^2 s^2 y s_2 \alpha \right. \]

\[ + 2h p_2 c b y c \beta - h^2 s^2 y s_2 \alpha + h^2 s y s c \alpha + h^2 c y s \alpha + 2 h s y s b c \alpha + k_1 (2 z \beta + z s_2 b c \alpha) \]

\[ -2 g h s \beta y c \alpha - g h c y s \alpha - g h s b y c \alpha \]

\[ l_1 = [+2 h c b y s \alpha 2 p_2 - 2 h s \beta c y] \]
\[m_2 = \left[ + \frac{h^2}{2} \gamma c \alpha s \beta - \frac{h^2}{4} s^2 \alpha s 2 \gamma c \beta + \frac{h^2}{4} s^2 \gamma c 2 \theta s \alpha s 2 \beta - \frac{h^2}{2} c^2 \gamma s 2 \beta + \frac{h^2}{4} s^2 - \frac{h^2}{2} s^2 \gamma s^2 a s 2 \beta \]
\[\quad - \frac{h^2}{2} c a c^2 \gamma s \beta - \frac{h^2}{4} s \alpha s 2 \gamma c \alpha 2 + \frac{h^2}{2} c^2 \gamma s 2 \beta - \frac{\sqrt{3}}{2} h^2 s \alpha s 2 \gamma c \alpha 2 + \frac{\sqrt{3}}{2} h^2 s^2 \gamma s 2 \beta + h_p z s a s y s \beta \]
\[\quad + h_p z c y c \beta - \frac{\sqrt{3}}{4} h^2 c^2 \gamma s 2 a c \beta + \frac{\sqrt{3}}{4} h^2 s \gamma s 2 a s 2 \beta + \frac{\sqrt{3}}{2} h^2 s a c^2 \gamma c \alpha 2 - \frac{\sqrt{3}}{2} h^2 s^2 \gamma c \alpha s \beta \]
\[\quad - \frac{g h}{2} s a s y c \beta + \frac{1}{2} g h c y s \beta + \frac{g h}{2} s a s y c \beta - \frac{g h}{2} c y s \beta - \frac{\sqrt{3}}{2} h_p z s a c y s \beta + \frac{\sqrt{3}}{2} h_p z s y c \beta \]
\[\quad + k (2 z s \alpha + z s 2 a c \beta) + \frac{\sqrt{3}}{2} g h s a c y c \beta\]

\[n_2 = \left[ + \frac{h^2}{2} \gamma c \beta \alpha s - \frac{h^2}{4} s \gamma c 2 a s \beta - \frac{h^2}{8} s^2 \beta s 2 \gamma c \alpha - \frac{h^2}{4} \gamma c 2 \alpha s \beta - \frac{h^2}{4} s^2 \gamma \alpha s 2 \alpha - \frac{h^2}{4} s^2 \beta s 2 \gamma s 2 \alpha \right] \]
\[\quad - \frac{h^2}{4} s^2 \alpha \gamma c s \beta - \frac{h^2}{8} s^2 \beta s 2 \gamma c \alpha - \frac{\sqrt{3}}{4} h^2 s^2 \gamma c \beta \alpha s - \frac{\sqrt{3}}{4} h^2 s^2 \beta s 2 \gamma c \alpha - h c \beta s y c a p z \]
\[\quad - \frac{h^2}{2} s^2 \gamma c \alpha + \frac{\sqrt{3}}{4} h c \beta c e a c p z + \frac{h^2}{4} h^2 s^2 \gamma c \alpha s - \frac{\sqrt{3}}{2} h^2 s^2 \beta c e a c \gamma + \frac{\sqrt{3}}{4} h^2 s^2 \gamma s 2 a s 2 \beta \]
\[\quad + \frac{\sqrt{3}}{2} g h c \beta c y c \alpha - \frac{g h}{2} s \beta \gamma s \alpha - \frac{\sqrt{3}}{2} g h s y s \alpha + \frac{g h}{2} c y s \alpha + k (2 z s \beta + z s 2 \beta c \alpha) + \frac{3}{2} g h c y s \alpha \]

\[l_2 = \left[ + \frac{\sqrt{3}}{4} h c \beta c y s \alpha + \frac{\sqrt{3}}{4} h s \beta s y - h c \beta s y s \alpha 2 p z + h s \beta c y \right]\]

\[m_3 = \left[ + \frac{h^2}{2} c^2 \gamma c \alpha s \beta + \frac{h^2}{2} s^2 \gamma c 2 \beta s \alpha + \frac{h^2}{4} s^2 \gamma s^2 a s 2 \beta - \frac{h^2}{4} c^2 \gamma s 2 \beta + h p z c c \beta \alpha - \frac{h^2}{2} s^2 \gamma s^2 a s 2 \beta \right] \]
\[\quad - \frac{h^2}{2} c a c^2 \gamma s \beta - \frac{h^2}{4} s \alpha s 2 \gamma c \alpha 2 - \frac{\sqrt{3}}{4} h^2 s a c^2 \gamma c \alpha 2 + \frac{\sqrt{3}}{4} h^2 s^2 \gamma c \alpha s \beta + \frac{\sqrt{3}}{4} h^2 s \alpha s 2 \gamma c \beta \]
\[\quad - \frac{\sqrt{3}}{4} h^2 s^2 \gamma s 2 \beta - \frac{g h}{2} c y s \beta + \frac{\sqrt{3}}{2} h p z s a c y s \beta - \frac{\sqrt{3}}{2} h p z s y c \beta + \frac{\sqrt{3}}{4} h^2 c^2 \gamma s 2 a c \beta \]
\[\quad + k_3 (2 z s \alpha + z s 2 a c \beta) - \frac{g h}{2} s a s y c \beta - \frac{g h}{2} s a s y c \beta + \frac{g h}{2} c y s \beta \]
\[ n_3 = \left[ -\frac{h^2}{4} s^2 \gamma s^2 \alpha + \frac{h^2}{4} s^2 \gamma c \beta s \alpha - \frac{h^2}{8} s^2 \beta s \gamma c \alpha - \frac{h^2}{4} s^2 \gamma s^2 \beta s \gamma c \alpha - \frac{h^2}{4} s^2 \gamma s^2 \beta s \gamma c \alpha \right. \\
\left. -\frac{\sqrt{3}}{4} h^2 s^2 \gamma s^2 \beta s \alpha + \frac{\sqrt{3}}{4} h^2 s^2 \gamma c \beta s \alpha + \frac{\sqrt{3}}{4} h^2 s^2 \beta s \gamma c \alpha + \frac{h}{2} c \gamma s \alpha + \frac{h}{2} s \beta s \gamma c \alpha \right] \\
-\sqrt{3} h c \beta \gamma c \alpha p_z - k_3(2 z s \beta + z s \beta c \alpha) + \frac{3}{2} g h c \gamma s \alpha - \frac{\sqrt{3}}{2} g h s \beta \gamma c \alpha \\
+ \frac{\sqrt{3}}{2} g h s \gamma s \alpha - \frac{g h}{2} s \beta s \gamma c \alpha + \sqrt{3} g h s \gamma s \alpha \\

l_3 = \left[ -\sqrt{3} h c \beta \gamma s \alpha - \sqrt{3} h s \beta s \gamma s \alpha - h s \beta c \gamma \right] \\
k_1 = \left[ -\frac{h^2}{4} c^2 s^2 s \alpha \beta + \frac{h^2}{4} s^2 \beta c^2 \gamma s \alpha - \frac{h^2}{4} c^2 \alpha s^2 \gamma - \frac{h^2}{4} s^2 \gamma s^2 \alpha \beta - \frac{h^2}{4} c^2 \beta s^2 \gamma + 2 h s \alpha s \beta c \gamma \\
-2 h c \beta s \gamma + h^2 s \alpha s \beta c \gamma - h^2 c \beta s \gamma + h^2 c^2 s \alpha s \gamma + 2 h s \beta s \gamma p_z - 2 g h s \alpha s \beta c \gamma + 2 g h c \beta s \gamma \\
-gh c \gamma s + g h s \alpha s \beta c \gamma + 2 h s \alpha c \beta c \gamma p_z \right]
\[ k_2 = \left[ + \frac{h^2}{4} c^2 \beta s 2 \gamma + h^2 c^2 a s 2 \gamma - \frac{h^2}{4} c^2 \gamma s 2 a s \beta + \frac{h^2}{4} s 2 \beta c 2 \gamma s a - \frac{h^2}{4} c^2 a s 2 \gamma + \frac{h^2}{4} s 2 \gamma s^2 a s^2 \beta \right. \\
\left. \frac{h^2}{4} c^2 \beta s 2 \gamma - h s \beta s y p_z + h^2 c^2 a s 2 \gamma + \frac{h^2}{4} s \beta s 2 \alpha c 2 \gamma - \frac{h^2}{4} s \alpha s 2 \beta c 2 \gamma - \frac{h^2}{4} s 2 \alpha c \gamma c \beta \right] \\
\left. - \frac{h^2}{4} s \alpha s 2 \beta c 2 \gamma - \frac{\sqrt{3}}{4} h^2 s \alpha s 2 \beta s 2 \gamma + \frac{\sqrt{3}}{4} h^2 c a c \beta c 2 \gamma - \frac{\sqrt{3}}{2} h^2 c^2 \beta c 2 \gamma - h s a c \beta c y p_z \right] \\
+ \frac{\sqrt{3}}{2} h^2 s a s 2 \beta c 2 \gamma + \frac{g h}{2} c a s y + \frac{g h}{2} s a s \beta c y - \frac{g h}{2} c \beta s y - \sqrt{3} h s a c \beta s y p_z + \sqrt{3} h s b c y p_z \\
\left. + \frac{\sqrt{3}}{4} h^2 s a s 2 \beta s 2 \gamma + \sqrt{3} g h c a c \gamma + \frac{3}{2} g h c a s y - \frac{\sqrt{3}}{2} g h s a s \beta s y - \frac{\sqrt{3}}{2} g h c \beta c y + \frac{\sqrt{3}}{2} g h c a c \gamma \right] \\
\left. - \frac{g h}{2} s a s \beta c y + \frac{g h}{2} c \beta s y \right] \\

\[ k_3 = \left[ - \frac{h^2}{4} c 2 \gamma s 2 a s \beta + \frac{h^2}{4} s 2 \beta c 2 \gamma s a - \frac{h^2}{4} c^2 a s 2 \gamma + \frac{h^2}{4} s 2 \gamma s^2 a s^2 \beta - \frac{h^2}{4} c^2 \beta s 2 \gamma - \frac{h^2}{4} s \alpha s 2 \beta c 2 \gamma \right. \\
\left. - \frac{h^2}{4} s 2 \gamma c a c \beta - \frac{h^2}{4} s \alpha s 2 \beta c 2 \gamma + \frac{h^2}{4} c^2 \beta s 2 \gamma + h^2 c^2 a s 2 \gamma + \frac{\sqrt{3}}{2} h^2 c^2 \beta c 2 \gamma - h s a c \beta c y p_z \right] \\
\left. - h s \beta s y p_z + h^2 c^2 a s 2 \gamma + \frac{h^2}{4} s \beta s 2 \alpha c 2 \gamma + \sqrt{3} h^2 c^2 a c 2 \gamma - \frac{\sqrt{3}}{4} h^2 s 2 a s \beta s 2 \gamma - \frac{\sqrt{3}}{2} h^2 s^2 a s^2 \beta c 2 \gamma \right] \\
\left. + \frac{\sqrt{3}}{4} h^2 s a s 2 \beta s 2 \gamma - \frac{\sqrt{3}}{2} h^2 c a c \beta c 2 \gamma - \sqrt{3} g h c a c \gamma + \frac{g h}{2} c a s y + \frac{g h}{2} s a s \beta c y - \frac{g h}{2} c \beta s y \right] \\
\left. + \sqrt{3} h s a c \beta s y p_z - \sqrt{3} h s b c y p_z + \frac{3}{2} g h c a s y + \frac{\sqrt{3}}{2} g h s a s \beta s y + \frac{\sqrt{3}}{2} g h c \beta c y - \frac{\sqrt{3}}{2} g h c a c \gamma \right] \\
\left. - \frac{g h}{2} s a s \beta c y + \frac{g h}{2} c \beta s y \right] \\
\]

\[ z = \frac{1}{(c a + c \beta)^2 + 4 s^2 a s^2 \beta} \]