SYNTHESIS OF PLANAR FOUR-BAR MECHANISMS

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ABSTRACT

Designing a four-bar mechanism that guides a coupler system through five given poses is an old and well known problem named after L. Burmester. In this paper we show that with kinematic mapping a much neater, more comprehensive solution is obtained. It produces a univariate quartic that can be solved explicitly. Furthermore, solutions that yield ordinary four-bars, slider-crank or elliptical trammels are identified, a-priori.

SYNTHESE DE MECANISMES PLAN A QUATRE BARRES.

RÉSUMÉ

La conception d'un mécanisme à quatre barres dirigeant un système de couplage à travers les cinq poses données soulève inévitablement le problème de Burmester. Dans notre contribution, nous démontrons que l'application cinématique permet d'obtenir une définition à la fois plus précise et plus complète. Elle génère un quartique à une variable qui peut être résolu de façon explicite. A priori, elle permettrait également l'identification de résolutions qui donnent à voir des mécanismes à quatre barres ordinaires, des mécanismes à manivelle et tiroir ou des doubles tiroirs.
1 INTRODUCTION

A planar four-bar mechanism is a closed kinematic chain, which consists of four bars, linked by four revolute joints. One link, called the base, is located in the fixed system $\Sigma_0$. It is connected with two links to the coupler, the moving system $\Sigma$. Given five finitely separated poses (position and orientation) $\Sigma_{t_1}, \ldots, \Sigma_{t_5}$ of $\Sigma$ (Fig.1) one can always find a finite set of planar four-bar mechanisms, guiding the coordinate system attached to the coupler through them. Note that not all poses necessarily have to lie in the same assembly branch of the four-bar. An algorithm to detect a branch defect only from the given poses (i.e. before synthesizing the mechanism) was presented recently in [15]. The problem of finding the design parameters of the four-bar when the five poses are given is called the five position Burmester problem, see Burmester [3]. This synthesis problem can be solved exactly due to the fact, that the five poses provide a number of equations equal to the number of variables. There exist a number of different ways to solve this problems, most of them use kinematic properties of the motion itself.

Bottema and Roth [2], McCarthy [13], Lichtenheld [12] and Hunt [9] solved the problem by intersecting the two center point curves to obtain the centers of the revolute joints in the fixed system. The centers of the revolute joints in the moving system are found by intersecting the two circle point curves. These points represent the points moving on circles in the synthesized four-bar motion. Bottema and Roth [2] also report on a solution of this problem in a 6-dimensional projective design parameter space by solving a system of two quadratic and four linear equations.

In this paper the problem is solved in closed form using the 3-dimensional projective kinematic image space of planar Euclidean displacements. Kinematic mapping was introduced independently by Blaschke [1] and Grünwald [5] in 1911. The first attempt to solve the Burmester problem using this method is published by Hayes and Zsombor-Murray [7]. Further developments are reported in Hayes et al.[8] and in the master theses of G. Qiao [14] and J. Nie [11]. Unfortunately in these papers only numerical solutions of the derived
set of algebraic equations is given. Furthermore the authors do not discuss the design of special four-bars like slider cranks and double sliders. To the best of the authors’ knowledge the paper at hand presents for the first time a complete and closed form solution of the Burmester problem using kinematic mapping, including a discussion of all special cases.

In kinematic mapping every displacement of the Euclidean plane is mapped to a point of a 3-dimensional kinematic image space. A one parameter motion is mapped to a curve, a two parameter motion is represented by a surface. As shown in Bottema and Roth [2], the constraint of a point bound to move on a circle with fixed center and radius maps to a hyperboloid of one sheet in the kinematic image space. If the circle degenerates to a line, the hyperboloid degenerates to a hyperbolic paraboloid as shown in Hayes and Husty [6]. In Bottema and Roth this hyperbolic paraboloid is called a special hyperboloid. Therefore the motion of the coupler, constrained by two points moving on circles, degenerate or not, is represented by the intersection of two hyperboloids of one sheet, of a hyperboloid of one sheet and a hyperbolic paraboloid or of two hyperbolic paraboloids in the kinematic image space. For the algebraic setup of the problem we follow Husty [10], who used the approach of Bottema and Roth [2].

The paper is organized as follows: In Section 2 we give a brief introduction to kinematic mapping of planar displacements. Section 3 deals with mechanism analysis. Section 4 establishes the constraint equations for the synthesis problem and presents conditions to distinguish, if among the synthesized mechanisms there exist special four-bars that pass through the five given poses. Furthermore in this section the solution of the synthesis problem in closed form is obtained. Section 5 gives an overview of the synthesis algorithm and Section 6 illustrates the algorithm with numerical examples.

2 PLANAR KINEMATIC MAPPING

Every Euclidean displacement \( d \) of a plane \( \Sigma \) can be written as \( p_0 = A \cdot p \), where \( p \) is a vector whose entries are the homogeneous coordinates of the moving point, expressed in \( \Sigma \), \( p_0 \) represents the same point in the fixed system \( \Sigma_0 \). The matrix \( A \) is given by

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{pmatrix},
\]

where \((a, b)\) are the components of the translation vector connecting the origins of \( \Sigma \) and \( \Sigma_0 \) and \( \phi \) is the rotation angle of \( \Sigma \) relative to \( \Sigma_0 \). In 1911 W. Blaschke and J. Grünwald simultaneously introduced kinematic mapping as a mapping \( \kappa \) of the planar Euclidean
displacements \((d \in SE_2)\) into a projective 3-space by

\[
\kappa : SE_2 \rightarrow P^3
\]

\[
d \mapsto \kappa(d) = \left(2 \cos \frac{\phi}{2} : 2 \sin \frac{\phi}{2} : a \sin \frac{\phi}{2} \right) = \left(X_0 : X_1 : X_2 : X_3\right).
\]

(2)

Every point in the kinematic image space, except those, where \(X_0 = X_1 = 0\), corresponds to a unique displacement of the plane. Given a point in the image space the entries of \(A\) can be computed with help of the following relations:

\[
\tan \frac{\phi}{2} = \frac{X_1}{X_0}, \quad a = \frac{2(X_1X_2 + X_0X_3)}{X_0^2 + X_1^2}, \quad b = \frac{2(X_1X_3 - X_0X_2)}{X_0^2 + X_1^2}.
\]

(3)

Using the expressions in Eq.3 \(p_0 = A \cdot p\) can be written with \(A\) in terms of the homogeneous image space coordinates:

\[
\begin{pmatrix}
Z \\
X \\
Y
\end{pmatrix} = \begin{pmatrix}
X_0^2 + X_1^2 & 0 & 0 \\
2(X_1X_2 + X_0X_3) & X_0^2 - X_1^2 & -2X_0X_1 \\
2(X_1X_3 - X_0X_2) & 2X_0X_1 & X_0^2 - X_1^2
\end{pmatrix} \begin{pmatrix}
z \\
x \\
y
\end{pmatrix}.
\]

(4)

In the following we will assume, that all points of interest in the moving system (coupler) are finite. Therefore we set \(z = 1\) in Eq. 4.
3 MECHANISM ANALYSIS

In this section the constraint equations representing the constraint of one point moving on a circle are derived. A circle in the fixed system $\Sigma_0$ is given by the equation:

$$C_0(X^2 + Y^2) - 2C_1XZ - 2C_2YZ + (C_1^2 + C_2^2 - R^2)Z^2 = 0 \quad (5)$$

where $(Z : X : Y)$ are the coordinates of the moving point on the circle, expressed in $\Sigma_0$; $(C_1, C_2)$ are the coordinates of the center and $R$ is the radius of the circle. The coefficient $C_0$ acts as a switch: if the circle degenerates to a line, $C_0 = 0$, else $C_0 = 1$. Substituting the values of $Z, X, Y$ from Eq.4 into Eq.5 yields the equation of a surface representing the constraint in the kinematic image space (Bottema and Roth [2]):

$$(R^2 - C_1^2 - C_2^2 - C_0(x^2 + y^2)) + 2C_1x + 2C_2y)X_0^2$$
$$+(R^2 - C_1^2 - C_2^2 - C_0(x^2 + y^2)) - 2C_1x - 2C_2y)X_1^2$$
$$+[(4C_2x - 4C_1y)X_1 + (4C_0y - 4C_2)x_2 + (-4C_0x + 4C_1)x_3]x_0$$
$$+[(4C_1 + 4C_2)x_2 + (4C_0y + 4C_2)x_3]x_1 - 4C_0x_2 - 4C_0x_2^2 = 0 \quad (6)$$

Given the constants $C_1, C_2, R, x, y$ and $C_0 = 1$, this quadric surface is a hyperboloid of one sheet, if $C_0 = 0$ the zero set of Eq.6 represents a hyperbolic paraboloid in the kinematic image space.

4 MECHANISM SYNTHESIS

In the Burmester synthesis problem five poses of a moving system $\Sigma$ are given. Without loss of generality we can assume that the fixed system $\Sigma_0$ coincides with one of these poses$^1$. Thus, the image space point, which represents the identity

$$(X_0 : X_1 : X_2 : X_3) = (1 : 0 : 0 : 0) \quad (7)$$

has to be on the constraint quadric Eq.6. Substituting condition (7) into Eq.6 yields an equation for the radius $R$:

$$R^2 - C_1^2 - C_2^2 - C_0(x^2 + y^2) + 2C_1x + 2C_2y = 0. \quad (8)$$

Solving Eq.8 for $R$ and substituting into Eq.6 we obtain the simplified circle constrained equation which will be crucial for the following discussion:

$$(-X_0X_3x + X_0X_2y + X_1X_3x + X_3X_1y - X_2^2 - X_3^2)C_0 - X_0X_2C_2 + X_0X_3C_1$$
$$+X_0X_1xC_2 - X_2^2xC_1 + X_1X_2C_1 - X_0X_1yC_1 - X_2^2yC_2 + X_1X_3C_2 = 0 \quad (9)$$

For further simplification we can use the fact that $C_0$ acts like a switch as mentioned in Section 3. Therefore we can make the following distinction of cases:

$^1$Otherwise the application of a unique Euclidean transformation, which does not change the design of the mechanism, will produce this situation.
4.1 Case A: At Least One Point Moves on a Line

Due to the fact that in this case $C_0 = 0$ holds, the constraint equation Eq. 9 simplifies to

$$ -X_0 X_2 C_2 + X_0 X_3 C_1 + X_0 X_1 x C_2 - X_1^2 x C_1 + X_1 X_2 C_1 - X_2 X_1 y C_1 - X_1^2 y C_2 + X_1 X_3 C_2 = 0. \quad (10) $$

Substituting the image space coordinates of the remaining four poses into this equation yields a system of four bilinear equations in the unknowns $C_1, C_2, x, y$:

$$ [(-C_2 x + C_1 y) X_{ii} - C_1 X_{ii} + C_2 X_{ii}] X_{0i} + (C_2 y + C_1 x) X_{1i}^2 + (-C_2 X_{ii} - C_1 X_{2i}) X_{1i} = 0 \quad i = 1, \ldots, 4. \quad (11) $$

Geometrically interpreted, these equations represent four quadrics in the design parameter space. If the five given poses can be reached by a slider crank or a double slider mechanism, this system has to have at least one non-trivial solution for the unknowns $C_1, C_2, x$ and $y$. Note, that $(C_1, C_2)$ is the normal vector of the line $l$ on which the point with the coordinates $(x, y)$ is constrained to move. Therefore only the ratio of $C_1$ and $C_2$ is relevant. This implicates that the system is overconstrained.

4.1.1 Subcase A_1: $l$ Is Parallel to the X-Axis of $\Sigma_0$

In this case we have $C_1 = 0$ and without loss of generality we can assume $C_2 = 1$. The constraint equations further simplify to:

$$ -X_{1i} X_{0i} x + X_{1i}^2 y + X_{2i} X_{0i} - X_{3i} X_{1i} = 0, \quad i = 1, \ldots, 4. \quad (12) $$

Geometrically these equations represent four lines in $\Sigma$. The system (12) is overconstrained and an easy consideration shows that it has solutions iff the four lines intersect in a point. Therefore a planar four-bar mechanism with at least one P-joint, having the axis parallel to the X-axis of $\Sigma_0$ and guiding a rigid body through the given poses can only exist, when the four lines in the set of equations (12) are in a pencil of lines. Solving any two equations of this system for $x, y$ and back substituting the solutions into the other two equations yields two compatibility conditions $E_1$ and $E_2$:

$$ E_1 : \quad \left( \frac{-X_{12} (-X_{1} X_{02} X_{22} + X_{01} X_{22} - X_{11} X_{21} X_{12} - X_{12} X_{11} X_{12})}{X_{12} X_{11} (X_{01} X_{12} - X_{11} X_{02})} + X_{23} \right) X_{03} - X_{13} X_{33} = 0, \quad (13) $$

$$ E_2 : \quad \left( \frac{-X_{14} (-X_{1} X_{02} X_{22} + X_{01} X_{22} - X_{11} X_{21} X_{12} - X_{12} X_{11} X_{12})}{X_{11} X_{12} (X_{01} X_{12} - X_{11} X_{02})} + X_{24} \right) X_{04} - X_{34} X_{14} = 0. \quad (14) $$

\footnote{In Eq. 11 the image space coordinates of the $i$th pose were denoted by $(X_{1i} : X_{2i} : X_{3i} : X_{4i})$.}
4.1.2 Subcase $A_2$: $l$ Is Not Parallel to the X-Axis of $\Sigma_0$

In this case we can set $C_1 = 1$. This yields four constraint equations of the form:

$$
((C_2 x + y)X_{1i} - X_{3i} + C_2X_{2i})X_{0i} + (C_2 y + x)X_{1i}^2 + (-C_2X_{3i} - X_{2i})X_{1i} = 0
$$

$$
i = 1, \ldots, 4. \tag{15}
$$

Solving any two equations of the set linearly for $x$ and $y$ and substituting the solutions into one of the remaining equations one ends up with an equation of degree three in $C_2$, which can be factored. Two solutions of this equation are always complex ($C_2 = \pm i$), and do not correspond to real mechanisms. The other factor of the equation can be solved linearly for $C_2$. Substituting the solutions for $x, y$ and $C_2$ into the remaining fourth equation yields a compatibility condition $E_3$, which is displayed in the Appendix.

Remark 1. Geometrically subcases $A_1$ and $A_2$ are of course identical. They only have to be distinguished because of the choice of the fixed coordinate system. (see figures in Sections 6.2.1 and 6.2.2)

4.2 Case B: At Least One Point Moves on a Circle

This is the general case and due to the fact, that in this case we can set $C_0 = 1$, the constraint equation simplifies to

$$
((-C_1 y + C_2 x)X_1 + (y - C_2)X_2 + (C_1 - x)X_3)X_0 + (-C_2 y - C_1 x)X_1^2
+((x + C_1)X_2 + (y + C_2)X_3)X_1 - X_3^2 - X_2^2 = 0. \tag{16}
$$

Following Gfrerrer [4] we apply a coordinate transformation for further simplification of the equations and set:

$$
-x - C_1 = 2b_1 \quad \quad \quad y - C_2 = 2b_3

-y - C_2 = 2b_2 \quad \quad \quad -x + C_1 = 2b_4. \tag{17}
$$

Applying this linear coordinate transformation, the circle constraint equation rewrites to

$$
[(2b_1 b_3 + 2b_2 b_4)X_1 + 2b_3 X_2 + 2b_4 X_3]X_0 + (-b_1^2 - b_2^2 + b_3^2 + b_4^2)X_1^2
+(-2b_1 X_2 - 2b_2 X_3)X_1 - X_3^2 - X_2^2 = 0. \tag{18}
$$

Eq.18 is the most simplified version of the circle constrained equation and will be used to derive a closed form solution of the general synthesis problem. Substituting the image space coordinates $(X_{1i} : X_{2i} : X_{3i} : X_{4i})$ of the $i$th pose into this equation yields the following system of constraint equations:

$$
[(2b_1 b_3 + 2b_2 b_4)X_{1i} + 2b_3 X_{2i} + 2b_4 X_{3i}]X_{0i} + (-b_1^2 - b_2^2 + b_3^2 + b_4^2)X_{1i}^2
+(-2b_1 X_{2i} - 2b_2 X_{3i})X_{1i} - X_{3i}^2 - X_{2i}^2 = 0
$$

$$
i = 1, \ldots, 4. \tag{19}
$$
The quadratic terms of the unknowns in the constraint equations (19) can be eliminated by simple manipulations (multiplication of equations with constant factors and subtractions of different ones). This yields three bilinear equations in the unknowns \( b_1, b_2, b_3, b_4 \). Performing the same process once more, one can get rid of the bilinear terms and two equations linear in the unknowns remain. Solving these equations for two of the unknowns, e.g. \( b_1, b_2 \), gives these two as functions of the other unknowns. Taking one of the bilinear equations and one out of the system (19) (now dependent just on \( b_3, b_4 \)) and calculating the resultant yields an equation of degree four in e.g. \( b_4 \), which is too long to be displayed here. This equation can be solved in closed form (see L. Ferrari (1522-1565)). It has 0, 2 or 4 real roots for \( b_4 \). Substituting one of this values in the two equations used for the resultant yields two quadratic equations in \( b_3 \). Because of the fact, that these two equations are now redundant, we can eliminate \( b_3^2 \) to produce one linear equation in \( b_3 \) and solve for this unknown. We can use the same procedure to obtain one value of \( b_3 \) for each real root of \( b_4 \). As \( b_1 \) and \( b_2 \) are linear functions of \( b_3 \) and \( b_4 \) they can be calculated easily.

With the inverse coordinate transformation to Eq.17 we obtain for each solution \( b_1, \ldots, b_4 \) the coordinates of the center \((C_1, C_2)\), which is one of the Burmester points, and the coordinates of the point \((x, y)\) constrained to move on the circle.

5 SYNTHESIS ALGORITHM

Initially, five poses of the moving system \( \Sigma \) are given as points in the kinematic image space with homogeneous coordinates \((X_{0i} : X_{1i} : X_{2i} : X_{3i})\) for \( i = 1, \ldots, 5 \). To obtain the situation, that one of the given points is the origin, one has to apply a fixed coordinate transformation to all of the given points. Note that such a fixed coordinate transformation does not change the kinematic situation. In the following we assume that this transformation has been performed and we refer just to the remaining four given points in the kinematic image space \((X_{0i} : X_{1i} : X_{2i} : X_{3i})\) for \( i = 1, \ldots, 4 \). But one always has to keep in mind, that the identity is then the remaining fifth Burmester position.

To determine if all positions of a resulting mechanism lie in a common assembly mode, one applies the branching defect detection algorithm of Schröcker, Husty and McCarthy [15].

Afterwards we have to test if among the Burmester points we are searching for there are points at infinity. This means, that the corresponding point \((x, y)\) of the coupler moves on a degenerated circle, a line and the synthesized \( RR \) chain is a slider. Therefore we have to substitute the values of the four given image space points into the equations \( E_1 \) (Eq.13), \( E_2 \) (Eq.14) and \( E_3 \) (Eq.9 in the Appendix). The different possible cases are dealt with in the subsections below.

In a next step it is necessary to plug the input data into the general case Equation (19) to obtain the finite solutions.
5.1 $E_1, E_2$ and $E_3$ Are Satisfied

If all three conditions are satisfied we know that at least two Burmester points are at infinity. One of them is the point at infinity of the Y-Axis of the fixed frame $\Sigma_0$. Such a motion, realized by a double slider, is a so called cardan motion (elliptical trammel). It also can be modelled by rolling a circle $c$ in another circle $C$ having double radius. The center of $C$ is then the point of intersection $T$ of the two given lines $l_1, l_2$, representing the axes of the sliders. $c$ is on $T$ and on the two moving points $Q_1, Q_2$. It is well known that the path of every point of $c$ in the rolling motion is a line passing through $T$. One can replace the two given lines $(l_1, l_2)$ with any two other lines $(l_3, l_4)$ having the same point of intersection $T$, as long as the moving points $(Q_1, Q_2)$ and $(Q_3, Q_4)$ lie on the same circle $c$ (Fig.3). This verifies, that one can always find a pair of axes for the P-joints with one line parallel to the $X$-axes of $\Sigma_0$. It is easy to see that the center of $c$ moves on a circle with the same radius centered in $T$. Therefore, as shown in Wunderlich [16], a cardan motion can also be realized by a slider crank, combining one of the possible sliders with this mentioned circle. Note that $C$ and $c$ are the fixed and moving axodes of both motions. In this case one obtains infinitely many double sliders and slider cranks, realizing always the same motion.

5.2 Only $E_1$ and $E_2$ Are Satisfied

If this is the case only one Burmester point is a point at infinity. The axis of the corresponding slider is parallel to the $X$-axes of $\Sigma_0$. Due to the fact that this is only one out of four Burmester points we additionally have to do the computations described in Subsection (4.2), which yield four circles, but one of them has infinite radius and corresponds to the axis of the slider. Assuming that all solutions are real we obtain in this case three general four-bars and three slider cranks passing through the five given poses.
5.3 Only $E_3$ Is Satisfied

In this case the synthesis results in one slider with an axis not parallel to the $X$-axes of $\Sigma_0$. The other Burmester points can be found in the same way as in the subsection before. The geometric interpretation is analogous to the subsection before: assuming that all solutions are real we obtain in this case three general four-bars and three slider cranks passing through the five given poses.

5.4 None of the Conditions $E_1, E_2, E_3$ Is Satisfied

Neither $E_1$, $E_2$ nor $E_3$ is satisfied. This means that the synthesis will yield only finite Burmester points. In this case we obtain 4 RR-dyads, which can be combined to 6 mechanisms.

Note that in all cases listed above either 0, 1 or 6 mechanisms can be real (0, 2 or 4 roots of Eq.19 are real). This concludes the discussion of all possible different cases. In the following a flowchart of the synthesis algorithm is given.

Algorithm 1.

1. Initially five poses of a moving system $\Sigma$ are given.

2. Apply a coordinate transformation to all given poses such that one of the poses coincides with the fixed coordinate frame.

3. Now only four arbitrary poses of a moving system remain, given by their coordinates $(X_{0i} : X_{1i} : X_{2i} : X_{3i})$ for $i = 1, \ldots, 4$ in the kinematic image space (see Section 2).

4. To determine if all poses lie in the same assembly mode see Schröcker, Husty and McCarthy [15].

5. Substitute the coordinates of the poses in the Equations $E_1$ (Eq.13), $E_2$ (Eq.14) and $E_3$ (see Appendix) to determine if some of the four Burmester points we are searching for are at infinity.

- $E_1$, $E_2$ and $E_3$ are satisfied: at least two Burmester points are at infinity and one of them is the point at infinity of the $Y$-axis of the fixed frame.
- Only $E_1$ and $E_2$ are satisfied: only one Burmester point is a point at infinity. This point is the point at infinity of the $X$-axes of the fixed frame.
- Only $E_3$ is satisfied: only one Burmester point is a point at infinity. This point is not the point at infinity of the $X$-axes of the fixed frame.
- None of the conditions $E_1$, $E_2$ and $E_3$ is satisfied: none of the Burmester points is at infinity.

6. Substitute the input data into the general case equation Eq.(19) to obtain the finite solutions.
To verify the established theory we have designed several mechanisms and used them to obtain five input poses. The design parameters are listed in Tables 1, 4, 7 and 10. Then the given poses were taken as input to the algorithm described above to synthesize four-bars. Note that in any case the algorithm has to return the input mechanisms and other RR-chains or degenerate RR-chains. The results are presented in tables, which show the synthesized mechanisms, the five chosen poses, the two points moving on circles or lines. In the figures we show several resulting mechanisms with the paths of three points of the moving system during the coupler motion. We mention that the depicted coordinate frames have to be thought rigidly attached to the coupler. For sake of clearness of the figures this connection has been omitted.

### 6.1 General Four-Bar Mechanism

<table>
<thead>
<tr>
<th>Table 1: Design parameters of mechanism 1</th>
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</thead>
<tbody>
<tr>
<td>$C_0$</td>
</tr>
<tr>
<td>$C_1$</td>
</tr>
<tr>
<td>$C_2$</td>
</tr>
<tr>
<td>$x$</td>
</tr>
<tr>
<td>$y$</td>
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<table>
<thead>
<tr>
<th>Table 2: Given relative poses</th>
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<tbody>
<tr>
<td>pose 1</td>
</tr>
<tr>
<td>$a$</td>
</tr>
<tr>
<td>$b$</td>
</tr>
<tr>
<td>$\phi$</td>
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<table>
<thead>
<tr>
<th>Table 3: Obtained results</th>
</tr>
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<tbody>
<tr>
<td>solution 1</td>
</tr>
<tr>
<td>$C_0$</td>
</tr>
<tr>
<td>$C_1$</td>
</tr>
<tr>
<td>$C_2$</td>
</tr>
<tr>
<td>$x$</td>
</tr>
<tr>
<td>$y$</td>
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</tbody>
</table>

The solutions 2 and 3 are the parameters of the input mechanism. Note that the given poses yield four real solutions. Therefore one can synthesize out of the four $RR$-dyads six real four-bar mechanisms that guide the coupler system through the given five poses.

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3Note that in all examples one pose is the identity and therefore will not be listed in the tables.
6.2 Slider Cranks

6.2.1 1 of the Input Mechanism Is Parallel to the X-Axis of $\Sigma_0$

![Figure 4: Two out of the six real general four-bar mechanisms that guide $\Sigma$ through the given poses](image)

**Table 4: Design parameters of mechanism 2**

<table>
<thead>
<tr>
<th>$C_0$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$\phi$</th>
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<tbody>
<tr>
<td>0</td>
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**Table 5: Given relative poses**

<table>
<thead>
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<th>pose 1</th>
<th>pose 2</th>
<th>pose 3</th>
<th>pose 4</th>
</tr>
</thead>
<tbody>
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<td>$a$</td>
<td>1.250980</td>
<td>2.240787</td>
<td>3.002000</td>
</tr>
<tr>
<td>$b$</td>
<td>0.003722</td>
<td>0.048305</td>
<td>0.386680</td>
</tr>
<tr>
<td>$\theta$</td>
<td>-0.023335</td>
<td>-0.088330</td>
<td>-0.278997</td>
</tr>
</tbody>
</table>

**Table 6: Obtained results**

As one can see in Table 6: solutions 2 and 4 yield the given input slider crank mechanism. There are again four real solutions, only one solution corresponds to a slider. The other
solutions are cranks. Therefore one can design three slider cranks and three general four-bars that guide the coupler through the given five poses.

Figure 5: A slider crank and a general four-bar mechanism that guide Σ through the given poses

6.2.2 I of the Starting Mechanism Is Not Parallel to the X-Axis of Σ₀

Table 7: Design parameters of mechanism 3

<table>
<thead>
<tr>
<th>solution 1</th>
<th>solution 2</th>
<th>solution 3</th>
<th>solution 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₀</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C₁</td>
<td>1</td>
<td>1.000952</td>
<td>1.153186</td>
</tr>
<tr>
<td>C₂</td>
<td>-1</td>
<td>3.066680</td>
<td>3.112102</td>
</tr>
<tr>
<td>x</td>
<td>9.99999</td>
<td>1.000438</td>
<td>1.071432</td>
</tr>
<tr>
<td>y</td>
<td>2.99999</td>
<td>7.999951</td>
<td>7.393329</td>
</tr>
<tr>
<td>⇒ R</td>
<td>∞</td>
<td>4.999271</td>
<td>4.881911</td>
</tr>
</tbody>
</table>

Table 8: Given relative poses

<table>
<thead>
<tr>
<th>pose 1</th>
<th>pose 2</th>
<th>pose 3</th>
<th>pose 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>2.876339</td>
<td>6.140158</td>
<td>7.937402</td>
</tr>
<tr>
<td>b</td>
<td>-0.146268</td>
<td>-0.300979</td>
<td>-0.469344</td>
</tr>
<tr>
<td>φ</td>
<td>0.221168</td>
<td>0.450081</td>
<td>0.589363</td>
</tr>
</tbody>
</table>

Table 9: Obtained results

Apart of the fact, that the axis of the slider is not parallel to the X-axis of Σ₀, the results of this case are analogous to the results of the previous case.
6.3 Double Slider Mechanism

In this case we obtain three solutions (Tab.12): solutions 1 and 2 correspond to the input sliders. Solution number three is a crank. As expected, the center of the third solution is the point of intersection of the lines of the input mechanism, the moving point is the center of $c$.

![Figure 6: One of the three slider cranks](image)

<table>
<thead>
<tr>
<th>$C_0$</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$C_2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y$</td>
<td>8</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 10: Design parameters of mechanism 4

<table>
<thead>
<tr>
<th>pose 1</th>
<th>pose 2</th>
<th>pose 3</th>
<th>pose 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>2.855481</td>
<td>5.609023</td>
<td>8.102182</td>
</tr>
<tr>
<td>$b$</td>
<td>-2.600297</td>
<td>-4.364129</td>
<td>-3.351390</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.341855</td>
<td>0.680246</td>
<td>0.998733</td>
</tr>
</tbody>
</table>

Table 11: Given relative poses

<table>
<thead>
<tr>
<th>solution 1</th>
<th>solution 2</th>
<th>solution 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$C_1$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>1</td>
<td>-1.67568</td>
</tr>
<tr>
<td>$x$</td>
<td>8</td>
<td>8.88721</td>
</tr>
<tr>
<td>$y$</td>
<td>3</td>
<td>6.51333</td>
</tr>
</tbody>
</table>

$\Rightarrow R \ \
\infty \quad \infty \quad 3.535545$

Table 12: Obtained results
7 CONCLUSION

In this paper a complete and closed form solution of the five position Burmester problem using kinematic mapping was presented. As a new result we have found three equations that will determine if there are sliders among the RR-chains to be synthesized. In detail all possible combinations of sliders, elliptical trammels and general four-bars were classified and discussed. The theoretical results were illustrated by a number of numerical examples. The investigation of the geometrical structure of the design parameter space and the meaning of equations $E_1$, $E_2$ and $E_3$ is subject of further research.

8 ACKNOWLEDGMENT

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REFERENCES


9 APPENDIX

\( (X_{14}X_{04}X_{01}X_{02}X_{31}X_{23}X_{22} - X_{34}X_{14}X_{02}X_{13}X_{21}X_{21}X_{11} - X_{34}X_{04}X_{12}X_{12}X_{01}X_{23} - X_{34}X_{04}X_{12}X_{11}X_{12}X_{21}) = 0 \)